PROBLEMS IN VARIATIONS FOR PLANE TRANSONIC GAS FLOW

PMM Vol. 35, №3, 1971, pp. 499-511

A. A. OREL
(Saratov)

(Received February 1, 1971)

Two examples are considered of the direct problem of transonic gas flow in the formulation of F. I. Frankl': the problem of flow through a Laval nozzle with nearly parallel walls, and the problem of flow past a symmetric wedge-like profile in a sonic gas stream. Calculation of the flow in the first approximation is reduced to a boundary-value problem for a second-order equation of mixed type. The boundary-value problem is in turn transformed to a singular integral equation with kernel of Cauchy type. The solution of the equation is sought by a method of successive approximations.

A condition is found that permits determination of the flux in the case of flow through a Laval nozzle, or the coefficient for the singular solution in the case of sonic gas flow past a profile.

The presence of nonlinear terms in the equations of transonic gas flow leads to great difficulties in the solution of the direct problem (the problem of finding the flow in a region having given boundaries).

In the case of plane parallel flow it is possible to transform the nonlinear equations in the flow plane into linear ones in the plane of the velocity hodograph. However the boundary-value problem can be posed in the hodograph plane only for a limited number of flows, with boundaries on which a relation between the velocity components is known in advance.

A. A. Nikol'skii [1] proposed a method of solving the boundary-value problem with boundaries that differ only slightly from those for which the transformation into the hodograph plane is known. As a result one obtains a boundary-value problem with linear boundary conditions for a linear second-order differential equation of mixed type.

The method of A. A. Nikol'skii was further developed in the work of F. I. Frankl' [2, 3], where it was applied to the solution of the direct problem of the Laval nozzle. It was shown that the flux through a nozzle with given walls is essentially undetermined, and can be prescribed in an arbitrary way.

In his subsequent work [4, 5] F. I. Frankl' discovered an undetermined coefficient also in the problem of sonic gas flow past a profile. In this case the solution is determined only to within an arbitrary multiplicative constant at the point of hodograph plane corresponding to the region infinitely remote from the body.

In these same papers the premise was advanced that the solution of the direct problem of transonic flow must satisfy some supplementary conditions, having a physical basis, which serve for finding the undetermined parameters in the problem.

The question of uniqueness of the flux through a given Laval nozzle was considered also in [6], in which uniqueness of flux was proved under the condition of conservation of the asymptotic type of flow at the center of the nozzle.

1. Formulation of problem. Consider a known plane parallel transonic flow of ideal gas in a region D having boundaries d_1 and d_2. As d_1, d_2 we may choose the upper and lower walls of a Laval nozzle, or the upper and lower surfaces of a profile in the stream.

If the transformation of the curves d_1 and d_2 into the hodograph plane is known, the flow field can be calculated as the solution of the corresponding boundary-value problem for the Tricomi equation

$$\eta\psi_{\theta\theta} + \psi_{\eta\eta} = 0 \qquad (1.1)$$

Here ψ is the stream function, θ the angle of inclination of the velocity vector, and η the variable of Frankl' [7].

Following the paper [2], we formulate the direct problem of finding the flow in a region E with boundaries e_1 and e_2 that differ only slightly from d_1 and d_2.

Let the flow in region D be given by the relations $x = x_0(\theta, \eta)$, $y = y_0(\theta, \eta)$ and $\psi = \psi_0(\theta, \eta)$, where x, y are Cartesian coordinates in the physical plane. Then the corresponding equations for the unknown flow can be written in the form

$$x = x_0(\theta, \eta) + \delta x(\theta, \eta), \quad y = y_0(\theta, \eta) + \delta y(\theta, \eta)$$
$$\psi = \psi_0(\theta, \eta) + \delta\psi(\theta, \eta)$$

Here δ is the variational symbol.

As shown in [2], the flow in region E can be calculated as the solution of the following boundary-value problem in the (θ, η) plane: to find the solution of the equation

$$\chi_{\theta\theta} + \frac{\partial}{\partial\eta}\left(\frac{1}{\eta}\chi_\eta\right) = 0 \qquad (1.2)$$

assuming on the boundaries d_1, d_2 the values

$$\chi = -\rho w \delta n_1 \ \text{ on } d_1, \quad \chi = \rho w \delta n_2 + \delta q \ \text{ on } d_2 \qquad (1.3)$$

Here ρ is the density of the gas, w the speed, and δn_i are the distances between the curves d_i and e_i measured along the inner normal to d_i $(i = 1, 2)$. The quantity δq appears in the boundary conditions (1.3) only in the case of nozzle flow, and is equal to the difference in flux through nozzles E and D.

A proof of theorems of existence and uniqueness of the solution of the boundary-value problem (1.2), (1.3) can be found in [8, 9].

The quantities δx, δy and $\delta\psi$ are given by the equations

$$\delta x = \frac{\cos\theta}{\rho w}\chi_\theta + \frac{\sin\theta}{\rho(1 - M^2)}\chi_w, \qquad \delta y = \frac{\sin\theta}{\rho w}\chi_\theta - \frac{\cos\theta}{\rho(1 - M^2)}\chi_w$$
$$\delta\psi = \chi - \frac{w}{1 - M^2}\chi_w, \qquad M - \text{Mach number} \qquad (1.4)$$

The relations (1.4) are simplified in the transonic approximation, and take the form

$$\delta x = \rho_* a_* \chi_\theta, \quad \delta y = \rho_* a_* (\gamma + 1)^{-1/3}\eta^{-1}\chi_\eta, \quad \delta\psi = (\gamma + 1)^{-1/3}\eta^{-1}\chi_\eta \qquad (1.5)$$

Here ρ_* and a_* are the critical values of density and sound speed, and γ is the adiabatic exponent.

Let the flow in region D be represented by the outflow of a stream from an infinite symmetric container with straight walls that make a certain angle θ_0 with the axis of symmetry CO (Fig. 1). This is a classical problem that has been considered by many authors (see [10], for example). The mapping of region D into the (θ, η) hodograph plane is shown in Fig. 2. By virtue of symmetry we can restrict attention only to the upper half of the flow field.

The streamline CO passing through the axis is chosen as the boundary d_1, and the upper wall of the nozzle as d_2. On the line OB the flow reaches the speed of sound.

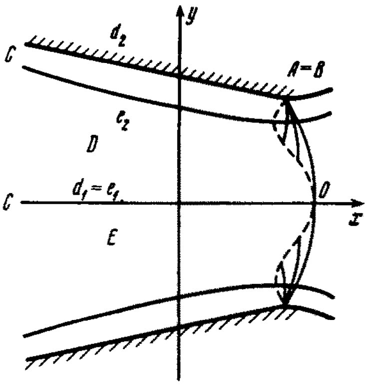

Fig. 1

In the region contained between the sonic line OB and the limiting characteristic OA appears an expansion flow with center at the point $A = B$, which is represented by the segment AB of a characteristic in the hodograph plane.

Calculation of the flow reduces to the boundary-value problem of Eq. (1.1) with the boundary conditions

$$\psi = 0 \text{ on } d_1, \ \psi = q \text{ on } d_2$$

We now consider the flow through the symmetrical Laval nozzle E (Fig. 1). We assume that the boundary e_1 coincides with d_1, and e_2 is given by the deviation δn from d_2. At the point $A = B$ the values of δn are measured in the direction of the normal to the velocity vector.

The flow in region E is calculated as the solution of the boundary-value problem (1.2), (1.3) with $\delta n_1 = 0$ and $\delta n_2 = \delta n$.

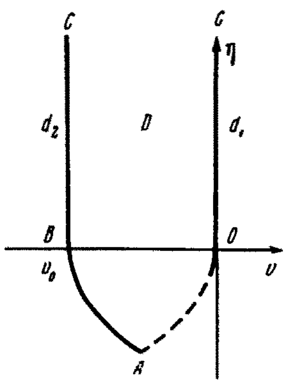

Fig. 2

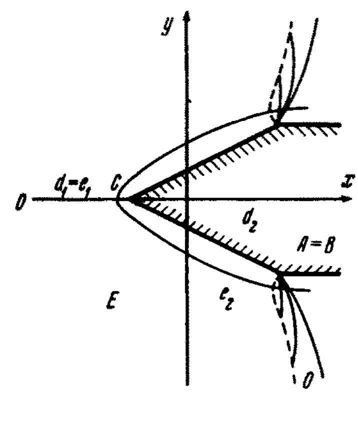

Fig. 3

We now consider the problem of symmetric flow of a sonic stream of gas past a wedge-like profile. We take as the basic flow that past a wedge at zero angle of attack (Fig. 3). The boundaries d_1 and d_2 are represented by the two branches of the single streamline passing through the axis of symmetry CO and the sides of the wedge. The image of region D in the hodograph plane coincides with that already considered in Fig. 2. The characteristic points in this flow are designated just as in the previous case. The semi-vertex angle of the wedge is equal to θ_0.

Calculation of the flow reduces to the determination of a function $\psi(\theta, \eta)$ that satisfies Eq. (1.1) and the boundary conditions $\psi = 0$ on d_1 and d_2. In addition, at the point O $(\theta = 0, \ \eta = 0)$ corresponding to infinity in the flow plane, the solution $\psi(\theta, \eta)$ must possess the singularity of Frankl' [10, 11]. In other words, the stream function for the flow in region D must have the form

$$\psi(\theta, \eta) = \psi_1(\theta, \eta) - f(\theta, \eta) \qquad (1.6)$$

where

$$f(\theta, \eta) = r^{-1/3}[(3t-1)(t+1)^{1/3} - (3t+1)(t-1)^{1/3}]$$

$$r^2 = \theta^2 + {}^4/_9\,\eta^3, \qquad t = \theta/r$$

The function ψ_1 is a bounded continuous solution of Eq. (1.1) and satisfies the following boundary conditions:

$$\psi_1 = 0 \quad \text{on} \quad d_1, \qquad \psi_1 = f(\theta, \eta) \text{ on } d_2$$

We choose the boundaries e_1, e_2 as shown in Fig. 3. The stream function for the flow in region E differs from ψ by the amount $\delta\psi$, which according to (1.6) must have the form $\delta\psi = \delta\psi_1 - \varepsilon f$. Hence, considering the relation between $\delta\psi$ and χ, we find

$$\chi(\theta, \eta) = \chi_1(\theta, \eta) - \delta c g(\theta, \eta)$$

where

$$g(\theta, \eta) = r^{-1/3}[(t+1)^{2/3} - (t-1)^{2/3}], \qquad \delta c = \text{const}$$

The problem is thus reduced to the determination of a continuous bounded solution of Eq. (1.2) that satisfies the boundary conditions

$$\chi_1 = 0 \text{ on } d_1, \qquad \chi_1 = \rho w \delta n + g\delta c \text{ on } d_2$$

The constant δc, like δq in the previous case, is an undetermined parameter of the problem. At the end of this paper we find a condition that permits δc or δq to be determined for a given δn.

2. Reduction to boundary-value problem with generalized equation. In Eq. (1.2) we transform to the new variables $x = 2\theta$, $y = \eta^2 \operatorname{sgn} \eta$ and $Z(x, y) = \chi(\theta, \eta)$. It then takes the form

$$\operatorname{sgn} y\,|y|^{-1/2} Z_{xx} + Z_{yy} = 0 \qquad (2.1)$$

We will, however, consider an equation of the more general form

$$\operatorname{sgn} y\,|y|^m Z_{xx} + Z_{yy} = 0 \qquad (-1 < m < +\infty) \qquad (2.2)$$

In the special case $m = 1$ we have the Tricomi equation (1.1); for $m = -{}^1/_2$ Eq. (2.2) agrees with (2.1).

Thus the examples under consideration of direct problems of transonic flow can, without loss of generality, be reduced to the following problem: to find a continuous bounded solution of Eq. (2.2) that satisfies the boundary conditions

$$Z(0, y) = \varphi_1(s)$$

$$Z(1, y) = \varphi_2(s) \qquad \left(s = \frac{2}{m+2} y^{1/2(m+2)}\right) \quad \text{for } y \geqslant 0 \qquad (2.3)$$

$$Z(x, y) = \psi(\mu) \text{ on the characteristic } \lambda = 0,\ 0 \leqslant \mu \leqslant 1$$

Here $\varphi_1(s)$, $\varphi_2(s)$ and $\psi(\mu)$ are given functions, and λ, μ are the characteristic coordinates

$$\lambda = x - \frac{2}{m+2}(-y)^{1/2(m+2)}, \qquad \mu = x + \frac{2}{m+2}(-y)^{1/2(m+2)}$$

Henceforth, to facilitate the calculation, we will assume that $\varphi_1(0) = \varphi_2(0) = \psi(0) = 0$. This assumption does not limit the generality of the reasonong, since instead of $Z(x, y)$ we can consider a solution of the form

$$Z(x, y) - [Z(1, 0) - Z(0, 0)]\, x - Z(0, 0)$$

On transition through the parabolic line $y = 0$ the condition must be satisfied that

$$Z_y(x, +0) = \operatorname{sgn} m \, Z_y(x, -0) \tag{2.4}$$

In the case $m = -{}^1/_2$ the relation (2.4) arises from continuity of the quantity $\delta\psi$ on transition through the sonic line, and for $m = 1$ it is the usual condition of Tricomi [12]. In addition, the derivative $Z_y(x, y)$ must be bounded in the flow region, since $\delta\psi$ is bounded in D.

3. Solution in elliptic region. Let the value of the derivative $Z_y(x, 0) = \nu(x)$ be known. We pose the following boundary-value problem in the elliptic plane: to find a solution of Eq. (2.2) that assumes in the half-strip D^+ $(0 \leqslant x \leqslant 1,\ y \geqslant 0)$ the boundary values

$$Z(0, y) = \varphi_1(s), \quad Z(1, y) = \varphi_2(s), \quad Z_y(x, 0) = \nu(x) \tag{3.1}$$

To construct the function $Z(x, y)$ we use a particular solution of Eq. (2.2) of the form

$$s^{1/2-\beta} J_{\beta-1/2}(s)\, e^{\pm x}, \qquad (x^2 + s^2)^{-\beta} \qquad \left(\beta = \frac{m}{2(m+2)}\right)$$

where $J_\nu(s)$ is the Bessel function of order ν.

We consider the expression

$$Z(x, y) = s^{1/2-\beta} \int_0^\infty J_{\beta-1/2}(ts)\, [a_1(t)\, e^{tx} + a_2(t)\, e^{-tx}]\, dt + \int_0^1 b(t)\, V(x, y; t)\, dt \tag{3.2}$$

$$V(x, y; t) = \sum_{n=-\infty}^{+\infty} \{[(2n + x - t)^2 + s^2]^{-\beta} - [(2n - x - t)^2 + s^2]^{-\beta}\}$$

The function $Z(x, y)$ is a solution of Eq. (2.2) for any $a_1(t)$, $a_2(t)$, $b(t)$ and satisfies the conditions

$$Z(0, y) = s^{1/2-\beta} \int_0^\infty J_{\beta-1/2}(ts)\, [a_1(t) + a_2(t)]\, dt$$

$$Z(1, y) = s^{1/2-\beta} \int_0^\infty J_{\beta-1/2}(ts)\, [a_1(t)\, e^{t} + a_2(t)\, e^{-t}]\, dt \tag{3.3}$$

$$Z_y(x, 0) = -\frac{b(x)}{k}, \qquad k = [2(1 - 2\beta)]^{2\beta} \frac{\Gamma^2(\beta)}{4\pi\Gamma(2\beta)}$$

Comparing expressions (3.1) and (3.3) and using the inversion formula for the Hankel integral transform, we find

$$a_1(t)\, e^{tx} + a_2(t)\, e^{-tx} = \frac{t \operatorname{sh} t(1 - x)}{\operatorname{sh} t} \int_0^\infty \varphi_1(\lambda)\, \lambda^{\beta+1/2} J_{\beta-1/2}(t\lambda)\, d\lambda +$$

$$+ \frac{t \operatorname{sh} tx}{\operatorname{sh} t} \int_0^\infty \varphi_2(\lambda)\, \lambda^{\beta+1/2} J_{\beta-1/2}(t\lambda)\, d\lambda, \qquad b(t) = -k\nu(t) \tag{3.4}$$

Substituting the relation (3.4) into (3.2) and interchanging the order of integration, we obtain

$$Z(x, y) = \int_0^\infty [\varphi_1(t) U(x, y; t) + \varphi_2(t) U(1 - x, y; t)]\, dt - k \int_0^1 \nu(t) V(x, y; t)\, dt \tag{3.5}$$

$$U(x, y; t) = s^{1/2-\beta} t^{\beta+1/2} \int_0^\infty \frac{\lambda \operatorname{sh} \lambda (1 - x)}{\operatorname{sh} \lambda} J_{\beta-1/2}(\lambda t) J_{\beta-1/2}(\lambda s)\, d\lambda \tag{3.6}$$

Let $Z(x, 0) = \tau(x)$; then from Eq. (3.5) we have

$$\tau(x) = -k \int_0^1 \nu(t) V(x, 0; t)\, dt + \Phi(x) \tag{3.7}$$

$$\Phi(x) = \int_0^\infty [\varphi_1(t) U(x, 0; t) + \varphi_2(t) U(1 - x, 0; t)]\, dt \tag{3.8}$$

We extend the definition of $\nu(x)$ as an odd periodic function in the interval $-\infty \leqslant x \leqslant +\infty$, and then the relation (3.7) can be written in the form

$$\tau(x) = -k \int_{-\infty}^{+\infty} \nu(t) |x - t|^{-2\beta} + \Phi(x) \tag{3.9}$$

In what follows the value of the derivative $\Phi'(0)$ is required. From Eqs. (3.8) and (3.6) we obtain

$$\Phi'(0) = \int_0^\infty [\varphi_1(t) U_1(t) - \varphi_2(t) U_2(t)]\, dt \tag{3.10}$$

$$U_1(t) = \lim_{x \to 0} U_x(x, 0; t) = -\frac{2^{1/2-\beta} t^{1/2+\beta}}{\Gamma(\beta + 1/2)} \int_0^\infty \lambda^{\beta+3/2} \operatorname{cth} \lambda J_{\beta-1/2}(\lambda t)\, d\lambda$$

$$U_2(t) = \lim_{x \to 1} U_x(x, 0; t) = -\frac{2^{1/2-\beta} t^{1/2+\beta}}{\Gamma(\beta + 1/2)} \int_0^\infty \frac{\lambda^{\beta+3/2}}{\operatorname{sh} \lambda} J_{\beta-1/2}(\lambda t)\, d\lambda$$

4. Solution in hyperbolic region. We consider the following problem in the hyperbolic half-plane: to find a solution of Eq. (2.2) valid in the characteristic triangle D^- ($\lambda = 0$, $\lambda = \mu$, $\mu = 1$) and satisfying conditions

$$Z(0, \mu) = \psi(\mu), \quad Z(0, 0) = 0, \quad \lim Z_y(\lambda, \mu) = \nu(x)$$

A suitable form of such a solution is found in paper [13] (see Chapter 5, Eqs. (4.15), (4.16) and (4.25)

$$Z(\lambda, \mu) = 2k \sin \pi\beta \int_0^\lambda \frac{\nu(\tau)\, d\tau}{[(\mu - \tau)(\lambda - \tau)]^\beta} + \frac{2 \cos \pi\beta}{\Gamma(1 - \beta)} \int_0^\lambda \frac{\Psi(\tau)\, d\tau}{[(\mu - \tau)(\lambda - \tau)]^\beta} +$$

$$+ \frac{1}{\Gamma(1 - \beta)} \int_\lambda^\mu \frac{\Psi(\tau)\, d\tau}{[(\mu - \tau)(\tau - \lambda)]^\beta}, \qquad \Psi(\tau) = \tau^\beta D^{1-\beta} \psi(\tau) \tag{4.1}$$

Here $D^\alpha f(x)$ is the integral of $f(x)$ of the fractional order α if $\alpha < 0$

$$D^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{f(t)\, dt}{(x - t)^{1+\alpha}}$$

or the derivative of the integral of $f(x)$ of fractional order α if $\alpha > 0$

$$D^{\alpha} f(x) = \frac{d^n}{dx^n} D^{\alpha - n} f(x) \qquad (n-1 < \alpha < n)$$

Equation (4.1) was obtained [13] for the case $-1/2 < m < 0$, but it is valid also for $m > 0$.

On the basis of (4.1) we can calculate the quantity

$$Z_y(\lambda, \mu) = [2(1-2\beta)]^{-2\beta} (\mu - \lambda)^{2\beta} (Z_\lambda - Z_\mu)$$

We introduce the operators

$$I_1[\lambda, \mu; f(\tau)] = (\mu - \lambda)^{1+2\beta} \int_0^{\lambda} \frac{f(\tau)\, d\tau}{[(\mu - \tau)(\lambda - \tau)]^{1+\beta}} \tag{4.2}$$

$$I_2[\lambda, \mu; f(\tau)] = (\mu - \lambda)^{1+2\beta} \int_{\lambda}^{\mu} \frac{f(\tau)\, d\tau}{[(\mu - \tau)(\tau - \lambda)]^{1+\beta}}$$

Then

$$Z_y(\lambda, \mu) = k_1 I_1[\lambda, \mu; \nu(\tau)] + k_2 \{2\cos\pi\beta I_1[\lambda, \mu; \Psi(\tau)] - I_2[\lambda, \mu; \Psi(\tau)]\}$$

$$k_1 = -\frac{\beta\Gamma^2(\beta)}{2\pi\Gamma(2\beta)} \sin\pi\beta, \qquad k_2 = \frac{k_1\Gamma(\beta)}{2\pi k} \tag{4.3}$$

If $f(\tau) = \tau^{\alpha}$, the integrals (4.2) are hypergeometric. Carrying out the substitution $\tau = \lambda t$ and correspondingly $\tau = \mu - (\mu - \lambda)t$, we find

$$\begin{aligned} I_1[\lambda, \mu; \tau^{\alpha}] &= B(1+\alpha, -\beta)\mu^{\beta}\lambda^{\alpha-\beta} F(-\beta, \alpha - 2\beta; 1 - \beta + \alpha; \lambda/\mu) \\ I_2[\lambda, \mu; \tau^{\alpha}] &= B(-\beta, -\beta)\mu^{\alpha} F(-\alpha, -\beta; -2\beta; 1 - \lambda/\mu) = \\ &= B(\beta - \alpha, -\beta)\mu^{\beta}\lambda^{\alpha-\beta} F(-\beta, \alpha - 2\beta; 1 - \beta + \alpha; \lambda/\mu) + \\ &+ B(\alpha - \beta, -\beta)\mu^{\alpha} F(-\beta, -\alpha; 1 + \beta - \alpha; \lambda/\mu) = \\ &= \sin\pi\alpha \csc[\pi(\alpha - \beta)] I_1[\lambda, \mu; \tau^{\alpha}] + B(\alpha - \beta, -\beta)\mu^{\alpha}[1 + O(\lambda/\mu)] \end{aligned} \tag{4.4}$$

Let $\psi''(\tau)$ be bounded and integrable in the interval (0, 1); then for $-1/2 < \beta < 0$ we obtain after simple transformations

$$\Psi(\tau) = \tau^{\beta} \frac{d^2}{d\tau^2} D^{-1-\beta}\psi(\tau) = \frac{\psi'(0)\tau^{2\beta}}{\Gamma(1+\beta)} + \tau^{\beta} D^{-1-\beta}\psi''(\tau) = c\tau^{2\beta} + O(\tau^{1+2\beta}) \tag{4.5}$$

The relations (4.4) and (4.5) show that $I_i[\lambda, \mu; \Psi(\tau)]$ $(i = 1, 2)$ is continuous and bounded in the region D^- except for the characteristic $\lambda = 0$, where it tends to infinity as λ^{β}. For $\alpha = 2\beta$ we have

$$\sin\pi\alpha \csc[\pi(\alpha - \beta)] = 2\cos\pi\beta$$

Therefore the expression in parentheses in Eq. (4.4) contains no terms with negative powers of λ, and is a continuous bounded function in the entire region D^-.

Thus if $\nu(x)$ is a continuous bounded function in the interval [0, 1], $Z_y(\lambda, \mu)$ is continuous and bounded in D^-, including the boundaries.

Let $y = 0$ ($\lambda = \mu = x$) and $Z(\lambda, \mu) = \tau(x)$. Then Eq. (4.1) assumes the form

$$\tau(x) = k_3 D^{2\beta-1} \nu(x) + G(x) \tag{4.6}$$

$$k_3 = 2k \sin \pi\beta \Gamma(1 - 2\beta), \qquad G(x) = \frac{2\Gamma(1+\beta)}{\Gamma(1+2\beta)} D^{2\beta-1} \Psi(x)$$

5. Reduction of boundary-value problem to singular integral equation. Comparing the expression for $\tau(x)$ from Eqs. (4.6) and (3.9), and also taking account of the condition (2.4), we obtain an equation for the determination of $\nu(x)$

$$D^{2\beta-1}\nu(x) + k_4 \int_{-\infty}^{+\infty} \nu(t) |x - t|^{-2\beta} dt = k_5 \varphi(x) \tag{5.1}$$

$$k_4 = \frac{\Gamma(2\beta)\cos \pi\beta}{\pi \operatorname{sgn} \beta}, \quad k_5 = \frac{k_4}{k}, \quad \varphi(x) = \Phi(x) - G(x)$$

We apply the operator $D^{1-2\beta}$ to the relation (5.1). Taking account of the fact that $D^{\alpha}D^{-\alpha} f(x) = f(x)$, and calculating the expression (cf. [13], Chapter V, Sect. 6)

$$D^{1-2\beta} \int_a^b f(t) |x - t|^{-2\beta} dt = \frac{\pi \operatorname{tg} \pi\beta}{\Gamma(2\beta)} f(x) + \frac{1}{\Gamma(2\beta)} \int_a^b \left(\frac{t}{x}\right)^{1-2\beta} \frac{f(t)\, dt}{t - x}$$

we obtain

$$\nu(x) + \lambda \int_{-\infty}^{+\infty} \left(\frac{t}{x}\right)^{1-2\beta} \frac{\nu(t)\, dt}{t - x} = g(x) \tag{5.2}$$

where

$$\lambda = \frac{\cos \pi\beta}{\pi(\operatorname{sgn} \beta + \sin \pi\beta)}, \quad g(x) = \mu D^{1-2\beta} \varphi(x), \quad \mu = \frac{\lambda\Gamma(2\beta)}{k}$$

We set $\nu(x) = x^{2\beta}\chi(x)$ and $g(x) = x^{2\beta} f(x)$, then by virtue of the periodicity of the function $\nu(x)$ we obtain from (5.2)

$$\chi(x) + \lambda \int_0^1 \chi(t) K(x, t)\, dt = f(x) \tag{5.3}$$

$$K(x, t) = \frac{1}{t - x} + \frac{1}{t + x} +$$

$$+ \sum_{n=1}^{\infty} \left[\left(\frac{t}{2n + t}\right)^{2\beta} \left(\frac{1}{2n + t + x} + \frac{1}{2n + t - x}\right) - \left(\frac{t}{2n - t}\right)^{2\beta} \left(\frac{1}{2n - t + x} + \frac{1}{2n - t - x}\right)\right]$$

Thus the solution of the boundary-value problem (2.2), (2.3), (2.5) is constructed if we succeed in finding a function $\chi(x)$ that satisfies the singular integral equation (5.3), with kernel of Cauchy type. We will solve Eq. (5.3) by the method proposed in [14] for the case when (2.2) is the Tricomi equation ($\beta = 1/_6$). We write Eq. (5.3) in the form

$$\chi(x) + \lambda \int_0^1 \chi(t) K_0(x, t)\, dt = r(x) \tag{5.4}$$

$$r(x) = f(x) - \lambda \int_0^1 \chi(t) \Delta K(x, t)\, dt, \quad \Delta K(x, t) = K(x, t) - K_0(x, t)$$

$$K_0(x, t) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2n+t+x} + \frac{1}{2n+t-x} \right)$$

Using the expansion of $\operatorname{ctg} x$ in elementary fractions, we represent $K_0(x, t)$ as

$$K_0(x, t) = {}^1/_2\pi \, [\operatorname{ctg} {}^1/_2\pi (t-x) + \operatorname{ctg} {}^1/_2\pi (t+x)] =$$
$$= \left(\sin^2 \frac{\pi}{2} t - \sin^2 \frac{\pi}{2} x \right)^{-1} \frac{d}{dt} \left(\sin^2 \frac{\pi}{2} t \right)$$

Substitution of the variables

$$y = \sin^2 \frac{\pi}{2} x, \quad \tau = \sin^2 \frac{\pi}{2} t, \quad \chi(x) = \mu(y), \quad r(x) = \psi(y) \tag{5.5}$$

permits (5.4) to be reduced to the characteristic equation

$$\mu(y) + \lambda \int_0^1 \frac{\mu(\tau)\, d\tau}{\tau - y} = \psi(y) \tag{5.6}$$

Applying the theory of singular integral equations [15], we write the solution of equation (5.6)

$$\mu(y) = \cos^2 \pi\theta \left\{ \psi(y) - \lambda \int_0^1 \left[\frac{(1-y)\tau}{(1-\tau) y} \right]^\theta \frac{\psi(\tau)\, d\tau}{\tau - y} \right\} + A y^{-1-\theta} (1-y)^\theta \tag{5.7}$$

$$\theta = -\pi^{-1} \operatorname{arctg} \pi\lambda = {}^1/_4 (2\beta - \operatorname{sgn} \beta), \qquad A = \text{const}$$

Restoring the previous variables (5.5) we obtain

$$\chi(x) = N[r(x)] = \cos^2 \pi\theta \left[r(x) - \lambda \int_0^1 \left(\frac{\operatorname{tg} {}^1/_2 \pi t}{\operatorname{tg} {}^1/_2 \pi x} \right)^{2\theta} K_0(x, t)\, r(t)\, dt \right] \tag{5.8}$$

The constant A from Eq. (5.7) is here chosen equal to zero in order that $\nu(x)$ be integrable in the interval [0, 1].

We introduce the operator

$$P[\chi(x)] = f(x) - \lambda \int_0^1 \chi(t)\, \Delta K(x, t)\, dt \tag{5.9}$$

Then the relation (5.8) is written in the form

$$\chi(x) = N[P[\chi(x)]] \tag{5.10}$$

From (5.10) we can find a Fredholm equation of the second kind for $\chi(x)$

$$\chi(x) + \lambda \int_0^1 \chi(t)\, \Gamma(x, t)\, dt = N[f(x)] \tag{5.11}$$

$$\Gamma(x, t) = N[\Delta K(x, t)] =$$
$$= \cos^2 \pi\theta \left[\Delta K(x, t) - \lambda \int_0^1 \left(\frac{\operatorname{tg} {}^1/_2 \pi\tau}{\operatorname{tg} {}^1/_2 \pi x} \right)^{2\theta} K_0(x, \tau)\, \Delta K(\tau, t)\, d\tau \right]$$

The kernel $\Gamma(x, t)$ has a complicated structure, so we determine $\chi(x)$ by applying a method of successive approximations. We determine the successive functions

$$\{\chi_n(x)\}_{n=1}^{\infty} \qquad (\chi_n(x) \in L_2(0, 1))$$

by means of the relations [14]

$$\chi_1(x) = N[P[0]] = N[f(x)], \quad \chi_n(x) = N[P[\chi_{n-1}(x)]] \quad (n = 2, 3 \ldots) \tag{5.12}$$

It can be shown that the sequence (5.12) converges to the solution of Eq. (5.11). From functional analysis [16] it is known that for the convergence of the sequence $\{\chi_n(x)\}_{n=1}^{\infty}$ it is necessary and sufficient that

$$\lim_{n,\, m\to\infty} \|\chi_n(x)-\chi_m(x)\| = 0$$

Here $\|f(x)\|$ is the norm of the function $f(x)$

$$\|f(x)\| = \left[\int_0^1 f^2(t)\,dt\right]^{1/2}$$

We form the expression

$$\chi_n(x)-\chi_{n-1}(x) = -\lambda\int_0^1 [\chi_{n-1}(t)-\chi_{n-2}(t)]\,\Gamma(x,t)\,dt =$$

$$= -\lambda N\int_0^1 [\chi_{n-1}(t)-\chi_{n-2}(t)]\,\Delta K(x,t)\,dt$$

Hence

$$\|\chi_n(x)-\chi_{n-1}(x)\| \leqslant a\,\|\chi_{n-1}(x)-\chi_{n-2}(x)\| \leqslant a^{n-1}\|\chi_1(x)\| \tag{5.13}$$

where

$$a = |\lambda|\,\|N[1]\|\max_{0\leqslant x,\, t\leqslant 1}|\Delta K(x,t)| \tag{5.14}$$

Assuming for definiteness that $n > m$, we find

$$\|\chi_n-\chi_m\| = \left\|\sum_{k=m+1}^{n}(\chi_k-\chi_{k-1})\right\| \leqslant \sum_{k=m+1}^{n}\|\chi_k-\chi_{k-1}\| \leqslant \sum_{k=m+1}^{\infty}\|\chi_k-\chi_{k-1}\| \tag{5.15}$$

Using (5.14) and (5.13) we can obtain the bound

$$\|\chi_n(x)-\chi_{n-1}(x)\| \leqslant \frac{a^m}{1-a}\|\chi_1(x)\|$$

Thus convergence of the sequence $\{\chi_n(x)\}_{n=1}^{\infty}$ is proved if we show that $a < 1$.

From Eq. (5.8) we have

$$N[1] = \cos^2\pi\theta\left[1-\lambda\int_0^1\left(\frac{\operatorname{tg}{}^1\!/_2\,\pi t}{\operatorname{tg}{}^1\!/_2\,\pi x}\right)^{2\theta}K_0(x,t)\,dt\right] =$$

$$= \cos^2\pi\theta\left\{1-\lambda\int_0^1\left[\frac{(1-y)\tau}{(1-\tau)y}\right]^{\theta}\frac{d\tau}{\tau-y}\right\} = \frac{\cos\pi\theta}{(\operatorname{tg}{}^1\!/_2\,\pi x)^{2\theta}}$$

The singular integral in the last expression is tabulated (see [17], Eq. 3.228.1). Knowing the function $N[1]$, we calculated its norm

$$\|N[1]\| = \cos\pi\theta\left[\int_0^1\frac{dt}{(\operatorname{tg}{}^1\!/_2\pi t)^{4\theta}}\right]^{1/2} = \frac{\cos\pi\theta}{(\cos 2\pi\theta)^{1/2}} \tag{5.16}$$

We consider the expression

$$\Delta K(x,t) = 2\sum_{n=1}^{\infty}\left\{\frac{(2n-t)^{1-2\beta}[(2n-t)^{2\beta}-t^{2\beta}]}{(2n-t)^2-x^2} - \frac{(2n+t)^{1-2\beta}[(2n+t)^{2\beta}-t^{2\beta}]}{(2n+t)^2-x^2}\right\}$$

It is not difficult to convince oneself that $\Delta K(x,t)$ is a continuous bounded function in the square $0\leqslant x,\ t\leqslant 1$, and also $\Delta K(x,0)=\Delta K(x,1)=0$, and $|\Delta K(x,t)|$ attains its maximum on the line $x=1$ at the point $t=1$

$$\max_{0\leqslant x,t\leqslant 1} |\Delta K(x, t)| = \lim_{t\to 1} |\Delta K(1, t)| = 4|\beta|, \qquad 0\leqslant x, t\leqslant 1 \qquad (5.17)$$

Equations (5.15), (5.16) and (5.17) permit calculation of the quantity a

$$a = 4|\lambda\beta| \frac{\cos\pi\theta}{(\cos 2\pi\theta)^{1/2}} = 2(1-4\theta)\frac{\sin\pi\theta}{\pi(\cos 2\pi\theta)^{1/2}}, \qquad 0\leqslant\theta\leqslant {}^1/_4$$

From the last relation it is easy to verify that $a < 1$. Consequently, the sequence (5.12) converges to the solution of Eq. (5.11).

In the first approximation for $\nu(x)$ we have the expression

$$\nu_1(x) = \cos^2\pi\theta \left\{g(x) - \frac{\lambda\pi}{2}\int_0^1 \left(\frac{x}{t}\right)^{2\beta}\left(\frac{\operatorname{tg}{}^1/_2\pi t}{\operatorname{tg}{}^1/_2\pi x}\right)^{\alpha}\left[\operatorname{ctg}\frac{\pi}{2}(t-x) + \right.\right.$$

$$\left.\left. + \operatorname{ctg}\frac{\pi}{2}(t+x)\right] g(t)\,dt\right\} \qquad \alpha = \beta - {}^1/_2\operatorname{sgn}\beta \qquad (5.18)$$

For $0 < \beta < {}^1/_2$ the function $\nu_1(x)$ is infinite of order ${}^1/_2 - \beta$ at the point $x = 1$. In the case $-{}^1/_2 < \beta < 0$ we have

$$g(x) = \mu D^{1-2\beta}\varphi(x) = \mu D^{1-2\beta}[\Phi(x) - G(x)] = \mu\left[D^{1-2\beta}\Phi(x) - \frac{2\Gamma(1+\beta)}{\Gamma(1+2\beta)}\Psi(x)\right]$$

Using the equation

$$D^{1-2\beta}\Phi(x) = \frac{\Phi'(0)}{\Gamma(1+2\beta)}x^{2\beta} + D^{-1-2\beta}\Phi''(x)$$

and the relation (4.5), we obtain

$$g(x) = \frac{\mu}{\Gamma(1+2\beta)}[\Phi'(0) - 2\psi'(0)]x^{2\beta} + O(x^{1+2\beta}) \qquad (5.19)$$

Thus for $g(x)$ to be bounded at the point $x = 0$, the condition must be satisfied that

$$\Phi'(0) - 2\psi'(0) = 0 \qquad (5.20)$$

The quantity $\Phi'(0)$ is given by Eq. (3.10). The singular integral in (5.18) can, for $-{}^1/_2 < \beta < 0$, be expressed in the form

$$2x^{\beta-1/2}\int_0^1 t^{-2\beta}\left(\operatorname{tg}\frac{\pi}{2}t\right)^{\beta-1/2} g(t)\,dt + O(x^{1/2+\beta})$$

it is bounded at the point $x = 0$ if

$$\int_0^1 t^{-2\beta}\left(\operatorname{tg}\frac{\pi}{2}t\right)^{\beta-1/2} g(t)\,dt = 0 \qquad (5.21)$$

Thus if the boundary values (2.3) satisfy the conditions (5.20) and (5.21), then $\nu(x) \in C[0, 1]$. From Eq. (4.3) it follows that in this case the function $Z_y(x, y)$ is bounded in the region of solution of the problem. Consequently, the quantity $\delta\psi$ is continuous and bounded in the flow region.

The relations (5.20) and (5.21) serve to determine the quantity δq or δc. At the same time they impose no restrictions on the form of the boundary e_2, since the latter is determined to within a shift relative to d_2 in the direction of the x-axis. In other words, $\delta n = \delta n'(x + \varepsilon)$. The constant ε is determined together with δq or δc from the conditions (5.20) and (5.21).

The author thanks S. V. Fal'kovich for guidance and continual interest.

BIBLIOGRAPHY

1. Nikol'skii, A. A., Equations in variations for plane adiabatic gas flow. In Collection of Theoretical Works on Aerodynamics. Moscow, Oborongiz, 1957.
2. Frankl', F. I., On the direct problem of the theory of the Laval nozzle. Academic Trans. Kabard.-Balkarsk. Univ., №3, 1959.
3. Frankl', F. I., Generalization of the problem of Tricomi and application to solution of the direct problem of the theory of the Laval nozzle. Academic Trans. Kabard.-Balkarsk. Univ., №3, 1959.
4. Frankl', F. I., Some questions of existence and uniqueness in transonic flow theory. Izv. Vuzov. Matem., №5, 1960.
5. Frankl', F. I., On the existence of a weak solution of the direct problem in the theory of sonic flow past a profile in the first approximation. Dokl. Akad. Nauk SSSR, Vol. 132, №4, 1960.
6. Lifshits, Iu. B. and Ryzhov, O. S., On the variation of gas flux in the calculation of the working regime of a Laval nozzle. J. U.S.S.R. Comput. Math. and Math. Phys., Pergamon Press, Vol. 6, №2, 1966.
7. Frankl', F. I., On the theory of the Laval nozzle. Izv. Akad. Nauk SSSR, Matem., Vol. 9, №5, 1945.
8. Frankl', F. I., Theorem of uniqueness for the solution of one boundary-value problem for the equation $u_{xx} + \partial/\partial y\,(u_y/y) = 0$. Izv. Vuzov. Matem., №1, 1959.
9. Frankl', F. I., Existence theorem for a weak solution of the direct problem in the theory of the plane-parallel Laval nozzle in the first approximation. Izv. Vuzov. Matem., №6, 1959.
10. Guderley, K. G., Theorie schallnaher Strömungen. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957.
11. Frankl', F. I., Investigation of the theory of the wing of infinite span moving at sonic speed. Dokl. Akad. Nauk SSSR, Vol. 57, №7, 1947.
12. Tricomi, F., Lectures on Partial Differential Equations. Moscow, Izd. inostr. lit., 1957.
13. Smirnov, M. M., Equations of Mixed Type. Moscow, "Nauka", 1970.
14. Sevost'ianov, G. D., Plane transonic gas flow reducing to the boundary-value problem of Tricomi for the half-strip. Collection "Transonic Gas Flow," Saratov, Saratov Univ., 1964.
15. Muskhelishvili, N. I., Singular Integral Equations, 2nd. Ed., Moscow, Fizmatgiz, 1962.
16. Natanson, I. P., Theory of Functions of a Real Variable. Moscow, Gostekhizdat, 1957.
17. Gradshtein, I. S. and Ryzhik, I. M., Tables of Integrals, Sums, Series, and Products, 4th ed., Moscow, Fizmatgiz, 1962.

Translated by M. D. V. D.